MATHEMATICS AT ADVANCED LEVEL

(OPTIONAL)

GRADE 9

2026-27

Academic Unit,

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INDEX

Note: Remaining four (04) chapters will be uploaded shortly.

Chapter 1

Sets

1.1 Introduction

Let's begin by taking some examples where we use the word "set" without having formally studied it. You must have heard about different blood types. The primary system used for blood typing is the ABO system. The four major blood types under this system are:

- A
- B
- AB
- O

Blood types have very important applications. If a person of blood type 'A' needs blood transfusion, then which **set of people** can donate blood?

OR

A teacher recommends a **set of books** in algebra to the student.

1.2 Set

So, we may talk about a set of people or a set of books in a casual manner. Since set is often used in mathematics, we define this term first.

Definition of a Set

A set is a well-defined collection of objects. The objects in a set are the elements or members of the set.

Three best students of your class is not a set, as best may vary for different individuals.

Some examples of set are:

(i) A set of students participating in a quiz contest.

(ii) A set of girls participating in Kho-Kho match.

On the other hand, collection of 3 most interesting books is not a set as different books may interest different students.

1.3 Representation of a Set

- **Roster Form:** One method of writing a set is to list all the elements of the set within braces separated by commas. For example, the set of vowels in English alphabet is written as V = $\{a, e, i, o, u\}$.

  Sets are denoted by capital letters.

  This method is also known as tabular form.

The fact that "e is an element of the set V" is written as $e \in$ V where the symbol '$\in$' means "belongs to".

Similarly, we use the symbol $c \notin$ V to denote c is not an element of the set V.

Example 1: Write the following sets in Roster form

(i) Set of whole numbers less than or equal to 5.

(ii) Set of first 4 terms of the A.P., whose first term is –3 and common difference is 4.

Solution: (i) A = {0, 1, 2, 3, 4, 5} (ii) B = {–3, 1, 5, 9}

Can all sets be written in roster form? Think about it!

- **Set Builder Form**

Another method of writing a set is set builder form. In this method we specify the elements of the given set by its description. All the elements of the set possess a single common property which no element outside the set possess.

For example, the set V = {a, e, i, o, u} can be written using this notation as

$$\{x \mid x \text{ is a vowel in English alphabet}\}$$

which is read as, "the set of all elements x such that x is a vowel of the English alphabet.

Example 2: Write the following sets in the set-builder form:

(i) $\{1, -1\}$ (ii) $\left\{\frac{2}{3}\right\}$ (iii) $\left\{\frac{1}{2}, \frac{1}{4}, 1, 2\right\}$

Solution: (i) $\{x \mid x \text{ is an integer and } x^2 = 1\}$

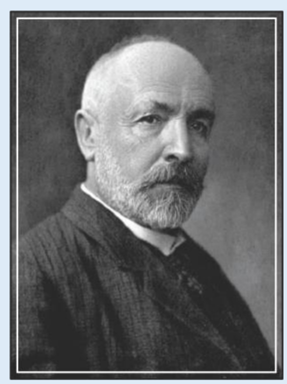
(ii) $\{x \mid x \text{ is fraction equivalent to } 0.\overline{6}\}$

(iii) $\left\{\frac{a}{b} \middle| a = 1 \text{ or } 4 \text{ and } b = 2 \text{ or } 4\right\}$

The theory of sets was developed by German mathematician Georg Cantor (1845-1918). He is considered the founder of set theory.

Cantor developed interest in mathematics in his teens. He began his university studies in Zurich but shifted to Berlin the very next year in 1863, where he studied under the eminent mathematician Karl Weierstrass, Ernst Kummer and Leopold Kronecker. He received his doctorate degree in 1867 in Number theory.

Cantor was also interested in philosophy and wrote papers relating his theory of sets to metaphysics. Cantor's contribution includes the discovery that the set of real numbers is uncountable. He is also noted for his contributions in analysis.



Georg Cantor
(1845-1918)

1.4 Finite and Infinite Sets

Can you make a list of students studying in your class? Of course it can be done easily, as the number of students are finite. Even the school's strength though, a large number, is finite. What about the set of natural numbers or set of real numbers between 2 and 3. Are they finite?

No these constitute infinite sets.

Some infinite sets are given below:

A = $\{x \mid x$ is a multiple of 5 and greater than 20$\}$

B = $\{x \mid x$ is a prime number$\}$

Example 3: State which of the following sets are finite or infinite:

(i) $\{x : x$ is an integer lying between 5 and 91$\}$

(ii) Set of coordinates of the points lying on a unit circle.

(iii) $\{x : x \in N$ and (x, y) satisfies the equation $2x + y = 8\}$

(iv) $\{x : x$ is the number of animals on earth$\}$

(v) $\{x : x$ is a digit in the decimal expansion of $\sqrt{2}\}$

Solution: (i) Since the set is $\{6, 7, 8, \ldots, 90\}$ so it is a finite set.

(ii) There are infinite points lying on a circle so its an infinite set.

(iii) Infinite points (x, y) where $x \in N$ lies on the line $2x + y = 8$. So set of values of abscissa forms an infinite set.

(iv) Set of animals on the earth is finite.

(v) $\sqrt{2}$ is an irrational number, so its decimal expansion is non-terminating, but the digits occurring in decimal expansion of $\sqrt{2}$ is finite.

Example 4: Write the following sets in roster form:

(i) $\{x \mid x$ is an odd natural number between 3 (excluding) and 11 (including)$\}$

(ii) $\{x \mid \sqrt{x}$ is a whole number less than or equal to 3$\}$

Solution: (i) $\{5, 7, 9, 11\}$ (ii) $\{0, 1, 4, 9\}$.

Example 5: Write the following sets in roster as well as set builder form.

(i) Real numbers between 2 and 5.

(ii) Fractions whose numerator and denominator are natural numbers and denominator exceeds the numerator by 1.

Is it possible to write in both the forms?

Solution: (i) Real numbers between 2 and 5.

Roster Form: Not possible to write the set in roster form, as we neither know its first element nor its last element. In fact, it's not possible to write two consecutive elements of this set.

Set Builder Form: $\{x : x \in R \text{ and } 2 < x < 5\}$

(ii) Fractions whose numerator and denominator are natural numbers and denominator exceeds the numerator by 1.

Roster Form: $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$

Set Builder Form: $\{x : x = \frac{n}{n+1}$, where $n \in$ natural number$\}$

Example 6: Use $\in$ or $\notin$ to indicate whether the given object is an element of the given set or not.

(i) $y \ldots \{a, x, y, z\}$

(ii) $4 \ldots \{1, 2, 5, 7, 11\}$

(iii) $2 \ldots \phi$

(iv) $5 \ldots \{x : x$ is a natural number $\leq 5\}$

Solution: (i) $\in$ (ii) $\notin$ (iii) $\notin$ (iv) $\in$

Example 7: Write each of the sets given below in the alternative form (roster or set builder)

(i) $\{2, 3, 5, 7, 11\}$

(ii) $\{1, 3, 5, 7, 9, 11\}$

Solution: (i) $\{x : x$ is a prime number less than $12\}$

(ii) $\{x : x$ is an odd natural number less than $13\}$

- **Is the order of elements in a set important?**

  The elements of a set distinguish the set – not the order in which the elements are written.

  Thus the three sets {1, 2, 3}, {1, 3, 2} and {3, 2, 1} are the same set.

- **Can a set have repeated elements?**

  Think about it!

  Try to form a set containing letters of the word "BANANA"

  Is it {B, A, N, A, N, A}? If it is correct, how many elements does the set contains, 6 or 3.

  In fact, repetition of elements is not allowed in a set. A set contains only distinct elements. So the set containing the letters of the word "BANANA" is {B, A, N}.

  Let us define some more type of sets:

1.5 Empty or Null Set

Consider the set {x : x is a prime number which is composite}

Is there a number which is both prime as well as composite?

Since no prime is composite, so the above set can be written in roster form as { }.

We say a set which does not contain any element is called an empty, null or void set. An empty set is denoted by the symbol ϕ or { }.

Example 8: Which of the following are empty sets:

(i) Set of even prime numbers.

(ii) Set of composite numbers having atmost 2 factors.

(iii) Set of numbers which are both rational and irrational.

(iv) Set of irrational numbers whose decimal expansion terminates.

Solution: (i) {2}; It is not an empty set.

(ii) Since a composite number has more than two factors, so it is an empty set.

(iii) Since no rational number is irrational, so set of numbers which are both rational and irrational is an empty set.

(iv) The decimal expansion of irrational number is neither terminating nor recurring. So it is an empty set.

1.6 Equality of Sets

Let A and B be two sets. We say A = B, if A and B have the same elements.

If two sets A and B are not equal, we write A $\neq$ B.

Some example of equal sets are:

{2, 5, –1, 0} and {0, –1, 5, 2} ;

{1, 2, 2, 2, 7, 7, 7, 7, 7} and {1, 2, 7} ;

{x : x is a prime divisor of 6} and {x : x is a pair of consecutive numbers that are prime}

EXERCISE 1.1

1. List the elements of the following sets :

(a) {x : x is an integer and $x^2 = 9$}

(b) {x : x is a positive integer less than 5}

(c) {x : x is even natural number divisible by 5}

(d) {x : $x \in$ N and $x < -1$}

2. Determine which elements of the set

$$A = \left\{-5, -\sqrt{3}, -\frac{1}{2}, 0, \frac{2}{5}, \pi, 13.4, \frac{1}{3}, \frac{19}{2}\right\}, \text{ are}$$

(a) Natural numbers
(b) Whole numbers
(c) Integers
(d) Rational numbers
(e) Real numbers

3. Write the following sets in roster form:

(a) $\{x : x$ is a two digit number and the sum of the digits is $5\}$
(b) $\{x : x$ is an integer and $|x| \geq 9\}$
(c) $\{x : x$ is letter of the word "SWEET"$\}$
(d) $\{x : x = \frac{n+1}{n}$, where n is a natural number and $n < 6\}$
(e) $\{x : x$ is a composite number$\}$

4. Write the following sets in set-builder form.

(i) $\{2, 4, 6, 8, \ldots\}$
(ii) $\{3, 6, 9, 12, 15\}$
(iii) $\{1, 4, 9, 16, \ldots\}$
(iv) $\{8, 9, 10, 11, \ldots\}$
(v) $\{1, 2, 3, 6\}$

Can two different sets have the same roster form?

5. Which of the following pairs of sets are equal.

(i) {D, E, C, E, N, T} and {C, E, N, T, D}
(ii) $\{a, b, \pi, \sqrt{2}\}$ and $\{a, \pi, \sqrt{2}, b\}$
(iii) $\{x : x$ is zero of the polynomial $x^2\}$ and $\{x : x$ is the root of the equation, $x^2 = 0\}$
(iv) $\{x : x$ has numerical value less than or equal to $1\}$ and $\{x : x$ is the root of the equation, $x^2 - 1 = 0\}$
(v) $\{5, 10, 15, 20\}$ and $\{5, 10, 15, 20, \ldots\}$
(vi) ϕ and $\{\phi\}$

6. State which of the following sets are finite or infinite.

(i) $\{x : x \in Z$ and $(x - 1)(x + 2)(x - 3) = 0\}$
(ii) $\{x : x$ and 2 are coprime$\}$
(iii) $\{x : x$ is a rational number between 3 and $4\}$
(iv) $\{x : x$ is an integer and $|x| \geq 5\}$

1.7 Subset

Consider the set X of students who live in the vicinity of 5 km radius around your school.

These students obviously along with others belongs to the set Y of all students in your class.

Since every student of set X belongs to or is contained in set Y, we say X is a subset of Y.

Definition: The set A is said to be a subset of B if and only if every element of A is also an element of B.

Symbolically, we write it as $A \subseteq B$.

If A is not a subset of set B, we write it as $A \not\subseteq B$.

We say $A \subseteq B$ if $a \in A \Rightarrow a \in B$.

Note that if A is a subset of B and $A \neq B$, then we say A is a proper subset of B and denote it by $A \subset B$.

- Is an empty set ϕ, a subset of a set containing at least one element, say P?

Yes, because there is no element in ϕ which is not in P.

Example 9: Use the notation $\subseteq$ to denote which set is the subset of the other in the following problems.

(i) $A = \{2, 3, p\}$ $\quad$ $B = \{1, 2, 3, p, q\}$

(ii) $A = \{2, 3, 5, 7\}$ $\quad$ $B = \phi$

(iii) $A = \{3, 8, 9, 0\}$ $\quad$ $B = \{0, 9, 8, 3\}$

(iv) $A = \{x \mid x = 2n, \text{ where } n \in N]$ $\quad$ $B = \{x \mid x = 4n, \text{ where } n \in N\}$

Solution:

(i) $A \subseteq B$

(ii) Since ϕ is subset of every set, So $B \subseteq A$

(iii) Since $A = B$, therefore each is a subset of the other i.e., $A \subseteq B$ as well as $B \subseteq A$.

(iv) $A = \{2, 4, 6, 8, 10, 12, \ldots\}$

$B = \{4, 8, 12, 16, 20, \ldots\}$

So $B \subseteq A$.

1.8 Cardinality of a Set

Let A be any set. If there are exactly m distinct elements in A, we say, cardinality of set A is m. Symbolically we write it as

$$n(A) = m, \text{ where } m \text{ is a non-negative integer,}$$

For example, if $A = \{1, 2, 3, \ldots, 9\}$ then $n(A) = 9$ and if $B = \left\{\pi, \sqrt{3}, 7\frac{1}{2}, 0\right\}$ then

$n(B) = 4$.

1.9 Power Set

Consider the set D = $\{a, b\}$. Can you write all its subsets?

Are $\{a\}$, $\{b\}$ its only subsets?

A little thinking would suggest some are still left.

It has two more subsets i.e. $\{a, b\}$ and ϕ.

So the set has 4 subsets in all i.e., $\{a\}$, $\{b\}$, $\{a, b\}$, ϕ.

If these subsets are written in a set form, we call it a power set of D.

i.e., $P(D) = \{\{a\}, \{b\}, \{a, b\}, \phi\}$

Note:
- $\{a\}$ and $\{a, b\}$ are elements of the set P(D).
- Neither a, nor a, b taken together are the elements of the set P(D).

Example 10: What is the power set of A = {0, 1, 2}?

Solution: Since the power set of A is again a set containing all its subsets, so

$P(A) = \{\phi, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$

Symbolically, we denote it by P(A).

Example 11: What is the power set of an empty set? What is the power set of $\{\phi\}$?

Solution: Since every set is a subset of itself. So ϕ is a subset of ϕ. Hence $P(\phi) = \{\phi\}$.

Also the set $\{\phi\}$ has exactly two subsets ϕ and $\{\phi\}$ itself hence $P(\{\phi\}) = \{\phi, \{\phi\}\}$.

A set containing n elements has 2^n subsets. If $n(A) = p$, where p is a whole number then $n[P(A)] = 2^p$.

Consider the set A = {1, 2} then it has 2^2 i.e., 4 subsets and $2^2 - 1 = 3$ proper subsets (i.e. ϕ, {1} and {2}).

1.10 Universal Set

In sets, the elements that we consider are usually limited to a specific all-encompassing set. For example, when we take sets of students belonging to a class or different sections of the same class or students of an editorial team, they all study in the same school. So the universal set denoted by 'U' are all the students of the school.

Similarly, if we discuss set of natural numbers, rational numbers and irrational numbers, then real numbers is the appropriate universal set. Universal set is denoted by U.

EXERCISE 1.2

1. Fill in the blanks with symbol $\subset$ or $\not\subset$.

   (i) {2, 3, 4} ………… {1, 2, 3, 4, 5}

   (ii) $\{x \mid x$ are triangles in a plane$\}$ ………… $\{x \mid x$ are polygons in a plane$\}$

(iii) {x is an integer} .... {$x : x$ is a multiple of 4}

(iv) ϕ .... {ϕ}

(v) $\{x \mid x = \frac{m-1}{m}$, where m is a non-zero integer} .... {x | x is a rational number}

(vi) $\{x \mid x = n^2\}$ .... $\{x \mid x = n^3\}$, (where n is a natural number)

(vii) $\{x \mid x \in R\}$ .... $\{x \mid x = 2^n\}$, (where R is a real number)

2. Determine whether the following statements are true or false.

(i) $1 \in \{1\}$

(ii) $\{2\} \in \{2\}$

(iii) $\{2\} \in \{\{2\}\}$

(iv) $\phi \in \{1, 2, 3\}$

(v) $\phi \subseteq \{1, 2, 3\}$

3. Write the power set of the following sets:

(i) $\{1\}$

(ii) $\{p, q\}$

(iii) $\{1, 2, 5\}$

(iv) $\{\phi, \{\phi\}\}$

4. What is the cardinality of the following sets:

(i) $\{a\}$

(ii) $\{a, \{a\}\}$

(iii) $\{\phi, 1, 2, \{1, 2\}\}$

(iv) $\{1, \{1\}, \{1, \{1\}\}\}$

(v) $\{\phi, \{\phi\}, \{\phi, \{\phi\}\}\}$

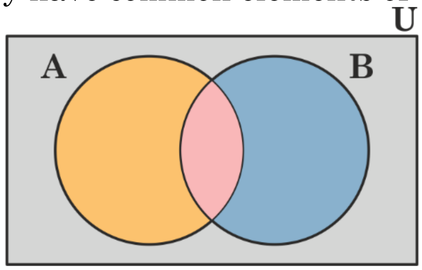
5. Let A be a set and $n(A) = 10$, then find the value of $n[P(A)]$? What if A has 100 elements?

1.11 Venn Diagram

A Venn diagram is a visual or pictorial representation of sets. This representation is known as Venn diagram after the English mathematician and philosopher John Venn (1834-1923). He studied in Cambridge, London. These diagrams are used to illustrate relationships between sets and draw logical results.

In the following figure two sets are represented by two circles enclosed in a rectangle.

The rectangle represents the universal set U. The circles may be overlapping or distinct depending upon they have common elements or not.



1.12 Set Operations

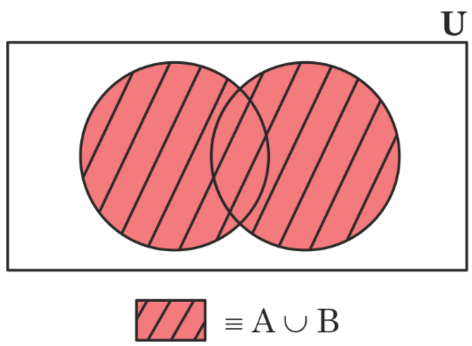
Two sets can be combined in many different ways. For instance, if we have set of students who play cricket and a set of students who play soccer, then we can form a set of students who play either cricket or soccer. We may also consider students who play both cricket and soccer. This is possible by performing certain operations on two sets to give another set.

Let us study these operations in detail.

1.12.1 Union of Two Sets

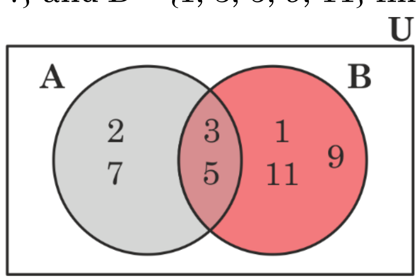
Let A and B be any two sets. Then union of A and B is the set containing elements of A or B or both A and B.

Symbolically, $A \cup B = \{x : x \in A \text{ or } x \in B\}$



Note: '$A \cup B$' is also denoted by A or B.

Example 12: If $A = \{2, 3, 5, 7\}$ and $B = \{1, 3, 5, 9, 11\}$ find $A \cup B$.



Solution: Since $A \cup B$ consists of all the elements of A as well as B.

Hence, $A \cup B = \{1, 2, 3, 5, 7, 9, 11\}$

Example 13: Find the union of the following pair of sets.

(i) $A = \{2, 5, 9\}$, $B = \{1, 4, 7\}$

(ii) $C = \{3, 5, 6\}$, $D = \{3, 4, 5, 6, 9\}$

(iii) $P = \{a, b, d, e\}$, $Q = \{b, c, e, f, g\}$

Solution:

(i) $A \cup B = \{1, 2, 4, 5, 7, 9\}$

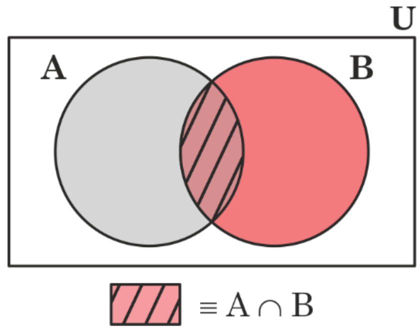
(ii) $C \cup D = \{3, 4, 5, 6, 9\}$

(iii) $P \cup Q = \{a, b, c, d, e, f, g\}$

1.12.2 Intersection of Two Sets

Let A and B be two sets. Then the intersection of A and B is the set that contains elements present in both A and B.

Symbolically, $A \cap B = \{x : x \in A \text{ and } x \in B\}$



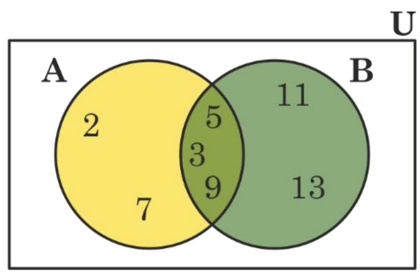
≡ $A \cap B$

Note: '$A \cap B$' is also denoted by 'A and B'.

Example 14: Find the intersection of the following two sets

$$A = \{2, 3, 5, 7, 9\}$$

$$B = \{3, 5, 9, 11, 13\}$$



Solution: Since $A \cap B$ consists of elements that are common to A and B.

Hence, $A \cap B = \{3, 5, 9\}$.

Example 15: Find the intersection of the following pairs of sets

(i) $\{a, b, f\}$ and $\{d, e, f, g\}$

(ii) $\{1, 2, 3, 6, 9\}$ and $\{1, 4, 5, 9, 13\}$

(iii) ϕ and $\{c, d, e\}$

(iv) $\{x : x$ is a natural number greater than 4 and less than 10$\}$ and $\{x : x$ is a factor of 12$\}$

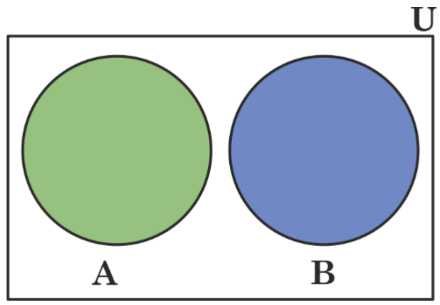
Solution: (i) $\{f\}$ (ii) $\{1, 9\}$ (iii) ϕ (iv) $\{6\}$

1.12.3 Disjoint Sets

Two sets A and B are disjoint if they have no element in common.

Therefore, the intersection of two disjoint sets is an empty set

$A \cap B = \phi$



Disjoint Sets

Example 16: Identify pair(s) of disjoint sets from the following:

$A = \{1, 3, 4\}$

$B = \{5, 6, 7, 8, \ldots\}$

$C = \{x : x \text{ is a prime factor of } 36\}$

$D = \{5, 7, 11, 13\}$

Solution: Since, $A \cap B = \phi$; A, B are disjoint

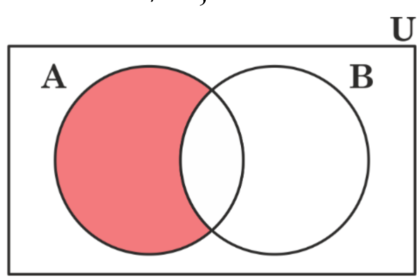
$A \cap D = \phi$; A, D are disjoint

$C \cap D = \phi$; C, D are disjoint.

1.12.4 Difference of Sets

Let A and B be two sets. The difference of A and B, denoted by A – B is the set containing elements which are in A but not in B.

Symbolically, $A - B = \{x : x \in A \text{ and } x \notin B\}$



$\equiv A - B$

The difference of {2, 5, 7} and {1, 2, 4} is {5, 7}.

Example 17: Find the difference of A and B from the following pair of sets.

(i) $A = \{1, 3, 5, 7, 9\}$, $B = \{2, 6, 8\}$

(ii) $A = \{a, b, p, q\}$, $B = \{b, q\}$

(iii) $A = \{1, 5, 9\}$, $B = \{1, 2, 4, 5, 7, 9\}$

Solution:

(i) $A - B = \{1, 3, 5, 7, 9\}$

(ii) $A - B = \{a, p\}$

(iii) $A - B = \phi$

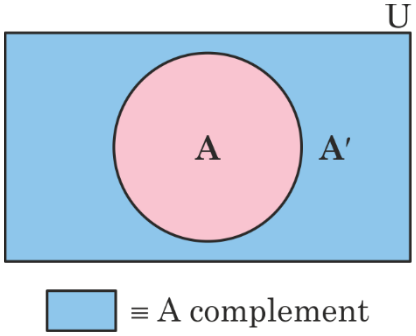
1.13 Complement of a Set

Let us learn complement of a set by taking a simple example.

Consider the students of your class as universal set. Let A be the set of boys in the class and B be the set of girls in the class. Then complement of boys is the set of students other than the boys i.e. the girls. Let us give the formal definition of the complement.

Definition of Complement

The complement of set A denoted by A′ is defined by $A' = \{x : x \in U \text{ and } x \notin A\}$



Example 18: Given,

$U = \{x \in N : x \le 9\}$

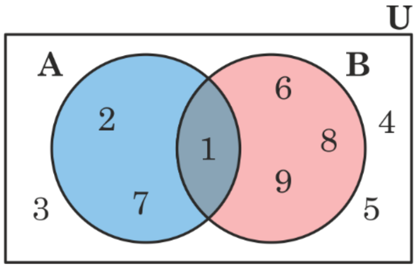
$A = \{1, 2, 7\}$

$B = \{1, 6, 8, 9\}$

find the following:

(a) A' (b) B' (c) $(A \cup B)'$ (d) $A' \cap B'$

Solution: $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$



(a) $A' = \{3, 4, 5, 6, 8, 9\}$

(b) $B' = \{2, 3, 4, 5, 7\}$

(c) $(A \cup B)' = \{1, 2, 6, 7, 8, 9\}' = \{3, 4, 5\}$

(d) $A' \cap B' = \{3, 4, 5, 6, 8, 9\} \cap \{2, 3, 4, 5, 7\} = \{3, 4, 5\}$

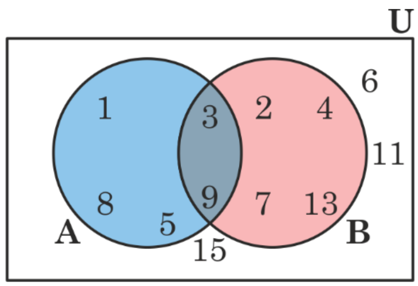
Note that $(A \cup B)' = A' \cap B'$. This law is known as **De-Morgan's Law**.

Example 19: Given A = {1, 3, 5, 8, 9}, B = {2, 3, 4, 7, 9, 13} and U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 15}

Use Venn diagram to verify the following De-Morgan's Laws.

(i) $(A \cap B)' = A' \cup B'$ (ii) $(A \cup B)' = A' \cap B'$

Solution: Representing given sets by Venn diagram we get



(i) $(A \cap B)' = U - (A \cap B) = \{1, 2, 4, 5, 6, 7, 8, 11, 13, 15\}$

$A' \cup B' = \{2, 4, 6, 7, 11, 13, 15\} \cup \{1, 5, 6, 8, 11, 15\}$

$= \{1, 2, 4, 5, 6, 7, 8, 11, 13, 15\}$

Hence, $(A \cap B)' = A' \cup B'$

(ii) $(A \cup B)' = U - (A \cup B) = \{6, 11, 15\}$

$A' \cap B' = \{2, 4, 6, 7, 11, 13, 15\} \cap \{1, 5, 6, 8, 11, 15\}$

$= \{6, 11, 15\}$

Hence, $(A \cup B)' = A' \cap B'$

1.14 Application of Sets

The concept of cardinal number finds many practical applications in real life. The theory of sets and the operations on them, provides some very useful formulae.

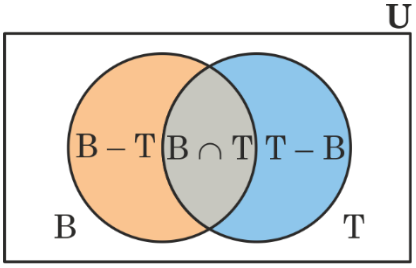
Let us now discuss and enlist a few observations which can be very easily verified using Venn diagrams.

1.14.1 If A and B are two finite sets, then their cardinal numbers are related as below:

1. n(Either in A or in B) = $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
2. n(Only in A, not in B) = $n(A - B) = n(A) - n(A \cap B)$
3. n(Neither in A nor in B) = $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$
4. n(Only in one of them) = $n[(A - B) \cup (B - A)] = n(A) + n(B) - 2\, n(A \cap B)$

Example 20: In a class with 40 students, 22 play badminton, 11 play both badminton and table tennis and 16 play neither badminton nor table tennis. How many play table tennis but not badminton?

Solution: Let B represents the set of students who play badminton and T represents the set of students who play table tennis.



We are given, $n(U) = 40$, $n(B) = 22$, $n(B \cap T) = 11$, $n(B \cup T)' = 16$

$$n(B \cup T) = n(U) - n(B \cup T)' = 40 - 16 = 24$$

Now, $$n(B \cup T) = n(B) + n(T) - n(B \cap T)$$

i.e., $$24 = 22 + P(T) - 11$$

or $$24 = 11 + P(T)$$

$$P(T) = 13 \text{ i.e. 13 students play table tennis}$$

n(Play table tennis but not badminton)

$$= n(T) - n(B \cap T)$$

$$= 13 - 11 = 2 \text{ students}$$

2 students play table tennis but not badminton.

Example 21: Each student from a group of 120 university students participated in teaching either a language or mathematics to the needy students. It is found that 92 of them can teach a language to the needy students and 46 can teach mathematics.

(a) Find the number of students who can teach both the subjects.

(b) Find the number of students who can teach only one of the two subjects.

Solution: If L denotes the set of students who can teach a language, M denotes the set of students who can teach mathematics

$n(L) = 92$, $n(M) = 46$ and $n(L \cup M) = 120$

Applying the relation

(a) $$n(\text{Either in L or in M}) = n(L \cup M)$$

$$= n(L) + n(M) - n(L \cap M) \text{ we get}$$

$$n(L \cap M) = n(L) + n(M) - n(L \cup M)$$

$$= 92 + 46 - 120$$

$$= 18$$

Therefore 18 students can teach both the subjects.

(b) n(Exactly in one of them) $= n[(L - M) \cup (M - L)]$

$= n(L) + n(M) - 2n(L \cap M)$

$= 92 + 46 - 2 \times 18$

$= 138 - 36$

$= 102$

Therefore 102 students can teach only one of the two subjects.

Note: You can also use the Venn diagram to solve this example.

Example 22: In a survey of 100 students regarding their preference for two subjects—Physics (P) and Chemistry (C), it is observed that 70 preferred Physics and 60 preferred Chemistry.

(i) Find the maximum possible number of students who neither like Physics nor Chemistry.

(ii) Find the minimum possible number of students who like both Physics and Chemistry.

Solution: We know that, $n(P \cup C) = n(P) + n(C) - n(P \cap C)$

(i) For maximum number of students who like neither of the subjects, we need $(P \cup C)'$

$n((P \cup C)') = n(U) - n(P \cup C)$ will be maximum if $n(P \cup C)$ is minimum. This happens when one of the two sets is a subset of the other, so that $P \cap C$ becomes the maximum.

The maximum possible value of $n(P \cap C)$ is the cardinal number of the smaller set, which is 60. (Means if everyone who likes Chemistry also likes Physics).

Thus, Minimum $n(P \cap C) = 70 + 60 - 60 = 70$.

Therefore, maximum number of students who neither like Physics nor Chemistry $= 100 - 70 = 30$

(ii) We know that $n(P \cup C) \leq n(U)$

This means $70 + 60 - n(P \cap C) \leq 100$

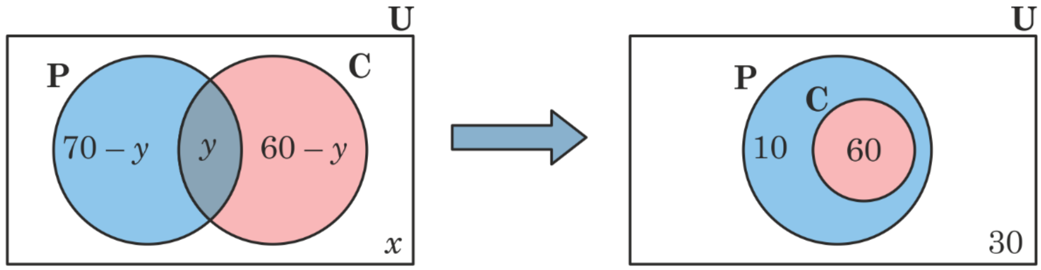
Therefore, $n(P \cap C) \geq 30$

Thus the minimum possible number of students who like both Physics and Chemistry is 30.

Alternate solution:

Let x be the number of students who like neither Physics nor Chemistry. These students sit outside the two circles but inside the rectangular Universal set (U).

So on the basis of information the following Venn diagram is drawn.



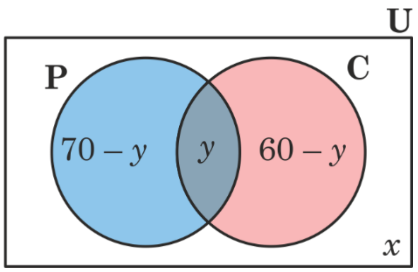
(i) Thus, to make x as large as possible, we must make the "space" occupied by the circles $(P \cup C)$ as small as possible. To make the union of two circles small, we must overlap them as much as possible. The most they can possibly overlap is the size of the smaller set. So, we put all 60 Chemistry students inside the Physics circle.

Therefore, Maximum value of neither Physics nor Chemistry = 30

(ii) Since $n(P \cup C) = n(P) + n(C) - n(P \cap C)$

$$n(P \cap C) = n(P) + n(C) - n(P \cup C)$$

Now $n(P \cap C)$ will be minimum if $n(P \cup C)$ is maximum.



The maximum possible value of $n(P \cup C) = 100$ (as there are a total of 100 students).

Hence, minimum value of $n(P \cap C)$

$$= n(P) + n(C) - n(P \cup C)$$

$$= 70 + 60 - 100 = 30$$

1.14.2 If A, B and C are three finite sets, then the relation between the cardinal numbers is given below:

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - [n(A \cap B) + n(B \cap C) + n(C \cap A)] + n(A \cap B \cap C)$$

Example 23: A market research group conducted an online survey of 1000 consumers and found that 35% of the consumers had rated a shampoo of type A by 5-stars. While 30% of the consumers rated the shampoo of type B by 5-stars and 250 consumers rated the shampoo of type C by 5-stars. It was observed that 200 consumers gave 5-star rating to the shampoos of both the types A and B, 150 consumers gave it to the shampoos of both the types B and C and 15% gave a 5-star to both A and C and 100 consumers rated all of them with 5-star.

(i) Find the percentage of consumers who gave 5-star rating to only one type of shampoo.

(ii) Find the number of consumers who did not give 5-star rating to any of the shampoos.

Solution: Let A, B and C represent the sets of consumers who rated 5-stars to the shampoos A, B and C respectively. We observe

$n(\text{U}) = 1000$; $n(\text{A}) = 30\%$ of $1000 = 350$;

$n(\text{B}) = 30\%$ of $1000 = 300$ and $n(\text{C}) = 250$

$n(\text{A} \cap \text{B}) = 200$; $n(\text{B} \cap \text{C}) = 150$;

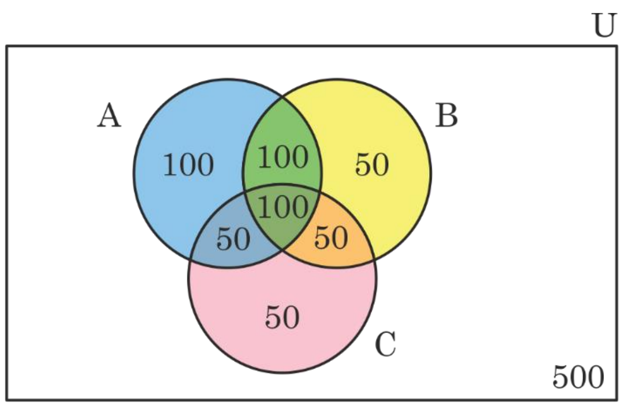
$n(\text{A} \cap \text{C}) = 15\%$ of $1000 = 150$ and $n(\text{A} \cap \text{B} \cap \text{C}) = 100$

Using Venn diagram we find:

$n(\text{only in A and B}) = n(\text{A} \cap \text{B}) - n(\text{A} \cap \text{B} \cap \text{C}) = 200 - 100 = 100$

$n(\text{only in B and C}) = n(\text{B} \cap \text{C}) - n(\text{A} \cap \text{B} \cap \text{C}) = 150 - 100 = 50$

$n(\text{only in A and C}) = n(\text{A} \cap \text{C}) - \text{n}(\text{A} \cap \text{B} \cap \text{C}) = 150 - 100 = 50$



(i) Thus, from Venn diagram, Number of consumers who gave 5-star rating to only one type of shampoo = 100 + 50 + 50 = 200

So, percentage of consumers who gave 5-star rating to only one type of shampoo

$$= \frac{200}{1000} \times 100 = 20\%$$

(ii) Now, number of consumers who did not give 5-star rating to any of the shampoos $n(\text{U}) - n(\text{A} \cup \text{B} \cup \text{C}) = 1000 - 500 = 500$

EXERCISE 1.3

1. Find the union of sets A and B i.e. $A \cup B$, in each of the following pairs.

(i) $A = \{1, 2, 3, 7\}$, $B = \{2, 7, 9\}$

(ii) $A = \{a, b, d, e\}$, $B = \{a, e, i, o, u\}$

(iii) $A = \{x : x \text{ is natural number} > 5\}$, $B = \{x : x \text{ is natural number} < 5\}$

(iv) $A = \phi$, $B = \{2, \sqrt{2}, -1, 0\}$

2. Evaluate each of the following.

(i) $\{1, 2\} \cap \{1, 2, 5\}$

(ii) $\{1, 3, 5, 7, 9\} \cap \{2, 4, 6, 8\}$

(iii) $\{g, o, a, t\} \cap \{c, a, t\}$

(iv) $\{x : x \text{ is an integer}\} \cap \{x : x \text{ is a negative integer}\}$

3. Which of the following sets are disjoint?

(i) $\{x : x \text{ is a multiple of } 2\}$ and $\{x : x \text{ is a multiple of } 3\}$

(ii) $\{e, \pi, \sqrt{2}, 0\}$ and $\left\{e^2, \frac{\pi}{2}, \sqrt{3}, 1\right\}$

(iii) $\{x : x \text{ is a real number}\}$ and $\{x : x \text{ is an irrational number}\}$

4. Find $A - B$ in each of the following.

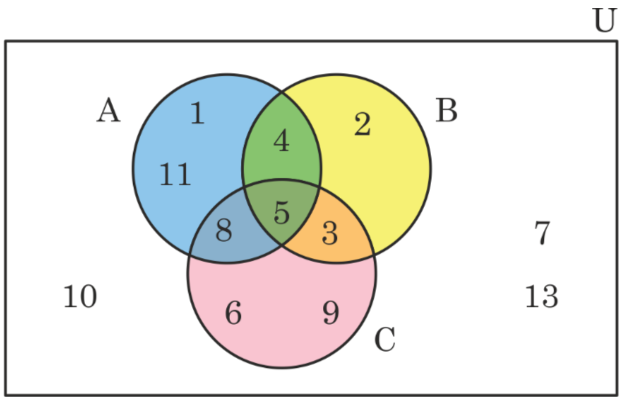
(i) $A = \{1, 3, 5, 8\}$, $B = \{3, 7, 8, 9\}$

(ii) $A = \{3, 0, 8\}$, $B = \{1, 3, 0, 8, 9\}$

(iii) $A = \{2, 6\}$, $B = \{1, 3, 5, 9\}$

5. Use the Venn diagram given below to answer the questions that follow.

Hint: You can find sets A, B, C and universal set U from the given Venn diagram.

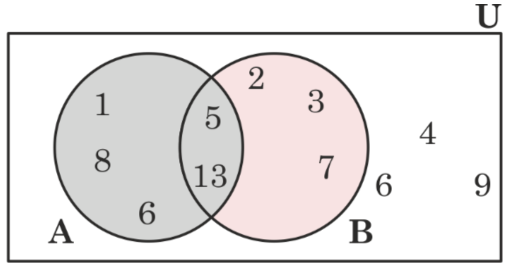


(i) A' (ii) B'

(iii) $(A \cap B)'$ (iv) $A' \cup B'$

(v) $A \cap B \cap C$ (vi) $A \cap (B \cup C)$

6. Verify $A - B = A \cap B'$ using the Venn diagram given below:



7. For a competitive exam, 85% of students opted for a Mock Test in Mathematics and 75% opted for a Mock Test in Science.

(a) What is the minimum possible percentage of students who opted for both tests?

(b) If 10% opted for neither, how does the minimum percentage for both change?

8. Let S be a set of 50 people. 35 people speak English and 25 speak Hindi. If k be the number of people who speak only English, then find the possible value of k, assuming every person speaks at least one of the two languages.

9. An organization awarded certificates to its 56 students for at least one of the three activities of Origami, Instrumental music and Fine arts. If 17 students received the certificates for Origami, 28 for Instrumental music, 25 for Fine arts and only 4 students got the certificates for all the three activities. Find the number of students who received the certificates for exactly two activities.

10. A survey of a group of 100 students in an international school revealed that 60 students could speak English, 50 students could speak German and 35 students could speak Spanish. Further 40 students could speak both English and German, 30 could speak both German and Spanish, 25 could speak both English and Spanish and 25 could speak all the three languages. Let E represent the set of students who speak English, G represents the set of students who speak German and S represent the set of students who speak Spanish. Answer the following using Venn diagram.

(a) How many students could speak at least two languages?

(b) How many students could speak at most one language?

(c) How many students could not speak any of the three languages?

Summary

(1) A set is a well-defined collection of objects.

(2) A set is represented in two forms

- Roster Form
- Set-Builder Form

(3) A set is finite if its number of elements is a natural number. Else it is infinite.

(4) A is subset of B if every element of A is contained in B. Symbolically it is denoted by $A \subseteq B$.

(5) **Cardinality** of a set is the number of distinct elements in it. It is denoted by $n(A)$.

(6) The set of all subsets of a set A is celled **power set** of A and is denoted by P(A).

(7) **Operation of sets.**

- **Union** of sets A and B is a set containing elements of A or elements of B or both. It is denoted by $A \cup B$.
- **Intersection** of sets A and B is a set containing elements of A and B. It is denoted by $A \cap B$.

(8) **Disjoint sets**

Two sets A and B are said to be disjoint if their intersection is an empty set

$$\text{i.e., } A \cap B = \phi$$

(9) **Difference** of two sets A and B is a set containing elements of A which are not in B

$$\text{i.e., } A - B = \{x : x \in A \text{ and } x \notin B\}$$

(10) **Complement** of a set A with respect to universal set U is a set containing the elements of U which are outside A

$$\text{i.e., } A' = U - A.$$

(11) **Applications of Sets** using **cardinal relations and Venn diagrams**

11.1 If A and B are two finite sets, then their cardinal numbers are related as below:

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A - B) = n(A) - n(A \cap B)$
- $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$
- $n((A - B) \cup (B - A)) = n(A) + n(B) - 2n(A \cap B)$

11.2 If A, B and C are three finite sets, then their cardinal numbers are related as follows:

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - [n(A \cap B) + n(B \cap C) + n(C \cap A)] + n(A \cap B \cap C)$$

Chapter 2

Logarithms

2.1 Introduction to Logarithms

Imagine a world before calculators and computers when mathematicians had to do complex calculations involving multiplication and division of large numbers. It took tremendous time and effort often involving lengthy calculations.

Opening Puzzle: The Sound of Numbers

In a music studio, the sound engineer says:

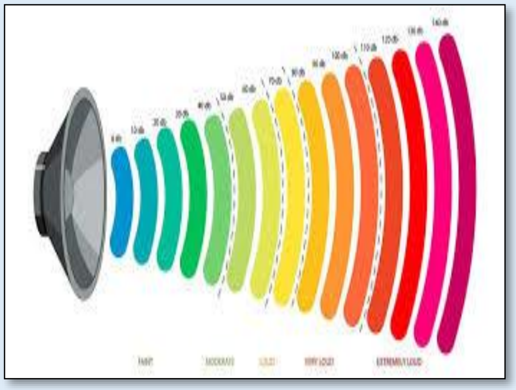
"This speaker produces a sound that is 1000 times more intense than the softest sound we can hear."

Instead of saying "1000 times," scientists say:

Sound Level = $\log_{10} 1000 = 3$

Why 3?

Because: $10^3 = 1000$

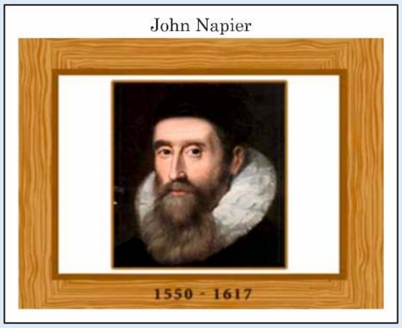


Then came **John Napier** with a revolutionary idea of effectively turning "multiplication into addition," and "division into subtraction" using logarithms. Logarithms is a tool that helps to do large calculations easily.

This approach saved scientists and fellow mathematicians a lot of tedious calculations.

Logarithms were introduced by John Napier (1550–1617), a Scottish mathematician. His method was different from the modern approach and was based on the relationship between arithmetic and geometric sequences. Later, Henry Briggs refined this idea and developed common logarithms (base 10), which made calculations easier.

Logarithm tables were widely used in science and engineering to simplify multiplication and division until electronic calculators became common. Even today, logarithms remain important in mathematics, especially natural logarithms (base *e*), which are widely used in calculus.



2.2 Understanding Logarithms as the Inverse of Exponents

We have earlier learnt about squares and cubes. For example:

- $10^2 = 100$ (10 squared is 100)
- $10^3 = 1000$ (10 cubed is 1000)

In these cases, we have taken the **base** 10, their **exponents** as 2 and 3. We find the value of base raised to the given exponent. But what if we know the resultant value and the base, and we want to find the exponent?

This power or exponent to which the base is raised is called a **Logarithm**.

- **Exponential Form:** $10^x = 100$
- **Logarithmic Form:** $\log_{10} 100 = x$

We already know about **exponents**: $2^3 = 8$

This means:

- Base = 2
- Exponent = 3
- Resultant = 8

Let us ask a different question: **2 raised to what power gives 8?**

That power is 3. So we say that logarithm of 8 to the base 2 is 3.

2.2.1 Understanding Logarithms through powers of 10

Let us look at powers of 10:

Powers of 10	Expressed in logarithmic form:
$10^0 = 1$ $10^1 = 10$ $10^2 = 100$ $10^3 = 1000$ $10^4 = 10000$	$\log_{10} 1 = 0$ $\log_{10} 10 = 1$ $\log_{10} 100 = 2$ $\log_{10} 1000 = 3$ $\log_{10} 10000 = 4$
$10^{-4} = 0.0001$ $10^{-3} = 0.001$ $10^{-2} = 0.01$ $10^{-1} = 0.1$ $10^0 = 1$	$\log_{10} 0.0001 = -4$ $\log_{10} 0.001 = -3$ $\log_{10} 0.01 = -2$ $\log_{10} 0.1 = -1$ $\log_{10} 1 = 0$

Definition: For any positive number b (where $b > 0$ and $b \neq 1$) and a positive number a if $b^x = a$ then $\log_b a = x$. We read this as " logarithm of a to the base b is x" .

For example, $32 = 2^5$ we can write $5 = \log_2 32$. These two statements are **equivalent** and we indicate this by using the symbol $\Leftrightarrow$. We write: $32 = 2^5 \Leftrightarrow \log_2 32 = 5$.

In general, $b^x = a$ where $a, b > 0$ and $b \neq 1$ and $\log_b a = x$ are **equivalent** statements and we write

$$b^x = a \quad \Leftrightarrow \quad \log_b a = x$$

Example 1: Write an equivalent form:

(a) Logarithmic form for $9^{\frac{1}{2}} = 3$

(b) Exponential form of $\log_5 625 = 4$

Math Talk: Can we find logarithm of a negative number? Why or why not?

Solution:

(a) $9^{\frac{1}{2}} = 3 \quad \Leftrightarrow \quad \log_9 3 = \dfrac{1}{2}$

(b) $\log_5 625 = 4 \quad \Leftrightarrow \quad 5^4 = 625$

Example 2: Rewrite $2^0 = 1$ in logarithmic form:

Math Talk: Is this true for all bases?

Solution: $2^0 = 1 \quad \Leftrightarrow \quad \log_2 1 = 0$

EXERCISE 2.1

1. Write an equivalent logarithmic statement for:

(a) $5^3 = 125$ (b) $(2)^5 = 32$

(c) $(7)^{-1} = \dfrac{1}{7}$ (d) $(3)^{\frac{-1}{2}} = \dfrac{1}{\sqrt{3}}$

2. Write an equivalent exponential statement for:

(a) $\log_2 16 = 4$ (b) $\log_9 81 = 2$

(c) $\log_5 \sqrt{5} = \dfrac{1}{2}$ (d) $\log_2\left(\dfrac{1}{2}\right) = -1$

3. Find the value of

(a) $\log_{10} 1000$ (b) $\log_6 36$ (c) $\log_2 64$

2.3 Logarithms Properties

The definition and the assumptions of logarithm are as follows:

Definition of Logarithm:

For $a > 0$, $a \neq 1$ and $x > 0$,

If $y = \log_a x$, then $a^y = x$.

Equivalently, $\log_a x = y \Leftrightarrow a^y = x$.

Assumptions:

- $a > 0$, $a \neq 1$ (The Base)
- $M > 0$, $N > 0$ (Positive real number)

These conditions ensure all logarithms in the following proofs, are defined.

Properties of Logarithms

Product Rule

Statement: $\log_a(MN) = \log_a M + \log_a N$

Proof: Let $x = \log_a(MN)$, $y = \log_a M$, $z = \log_a N$

Then, $a^x = MN$, $a^y = M$ and $a^z = N$

Therefore, $a^{y+z} = a^y a^z = MN \Rightarrow y + z = \log_a(MN)$

Thus, $\log_a M + \log_a N = \log_a(MN)$

Quotient Rule

Statement: $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$

Proof: Let $x = \log_a\left(\frac{M}{N}\right)$, $y = \log_a M$, $z = \log_a N$

Then, $a^x = \left(\frac{M}{N}\right)$, $a^y = M$ and $a^z = N$

Therefore, $a^{y-z} = \frac{a^y}{a^z} = \left(\frac{M}{N}\right) \Rightarrow y - z = \log_a\left(\frac{M}{N}\right)$

Thus, $\log_a M - \log_a N = \log_a\left(\frac{M}{N}\right)$

Power Rule

Statement: $\log_a\left(M^k\right) = k\log_a M$

Proof: Let $x = \log_a\left(M^k\right)$ and $y = \log_a M$

Then, $a^x = M^k$, $a^y = M$

Therefore, $a^x = a^{ky} \Rightarrow x = ky$

Thus, $\log_a\left(M^k\right) = k\log_a M$

Change of Base Formula

Statement: $\log_a(M) = \dfrac{\log_b M}{\log_b(a)}$, For $b > 0$, $b \neq 1$

Proof: Let $x = \log_a(M) \Rightarrow a^x = M$

Now take log with base b on both the sides,

$\log_b(a^x) = \log_b M \Rightarrow x\log_b a = \log_b M$

Thus, $x = \log_a(M) = \dfrac{\log_b(M)}{\log_b(a)}$

Log of 1

Statement: $\log_a(1) = 0$

Proof: Let $x = \log_a 1 \Rightarrow a^x = 1$

Since, $a^0 = 1$, for any $a > 0$, $a \neq 1$

$a^0 = 1 = a^x \Rightarrow x = 0$

Thus, $x = \log_a(1) = 0$

Log of *a* number to the same base

Statement: $\log_a a = 1$

Proof: Let $x = \log_a a \Rightarrow a^x = a$

Since, $a^1 = a$, for any $a > 0$, $a \neq 1$

$a^1 = a^x \Rightarrow x = 1$

Thus, $x = \log_a a = 1$

Let us summarize the properties of logarithms we proved above in the table below:

For any base a, $a > 0$, $a \neq 1$ and M, N > 0

Property Name	Logarithmic Notation	What it does
Product Rule	$\log_a(\mathrm{M}\times\mathrm{N}) = \log_a \mathrm{M} + \log_a \mathrm{N}$	Turns multiplication into addition
Quotient Rule	$\log_a\left(\frac{\mathrm{M}}{\mathrm{N}}\right) = \log_a \mathrm{M} - \log_a \mathrm{N}$	Turns division into subtraction
Power Rule	$\log_a(\mathrm{M}^k) = k \times \log_a \mathrm{M}$	The exponent comes down
log of number to the same base	$\log_a(a) = 1$	Log of any number to the same base is 1
log of 1	$\log_a(1) = 0$	Logarithm of 1 to any base is zero
Base changing property	$\log_a n = \frac{\log_b n}{\log_b a}$	Base a is changed to any other base b ($b > 0$ and $b \neq 1$)

Remember: For positive values of m, n, x, y and $a > 0$, $a \neq 1$,

- $\log_a(m+n) \neq \log_a m + \log_a n$
- $\log_a(m-n) \neq \log_a m - \log_a n$
- If $x \neq y$ then $\log_a x \neq \log_a y$

Example 3: Write it as a single logarithm

(a) $\log_7 3 + \log_7 5$ (b) $\log_2 9 - \log_2 3$

(c) $\log_4 3 + \log_4 6 - 3\log_4 2$ (d) $1 + \log_3 5$

Solution:

(a) $\log_7 3 + \log_7 5 = \log_7(3\times 5)$

$= \log_7 15$

(b) $\log_2 9 - \log_2 3 = \log_2\left(\frac{9}{3}\right)$

$= \log_2 3$

(c) $\log_4 3 + \log_4 6 - 3\log_4 2 = \log_4(3\times 6) - \log_4 2^3$

$= \log_4(18) - \log_4 8$

$= \log_4\left(\frac{18}{8}\right)$

$= \log_4\left(\frac{9}{4}\right)$

(d) $1 + \log_3 5 = \log_3(3) + \log_3 5$

$= \log_3(3\times 5)$

$= \log_3 15$

Example 4: Find the value of:

(a) $\log_7 343$ (b) $\log_3 27\sqrt{3}$

Solution:

(a) $\log_7 343 = \log_7(7^3)$

$= 3\log_7(7) = 3$

(b) $\log_3 27\sqrt{3} = \log_3\left(3^3 \times 3^{\frac{1}{2}}\right)$

$= \log_3 3^{\left(3+\frac{1}{2}\right)}$

$= \log_3 3^{\left(\frac{7}{2}\right)} = \frac{7}{2}$

Example 5: Simplify: $\log_3 81 - \log_3 9$

Solution: $\log_3 81 - \log_3 9 = \log_3 3^4 - \log_3 3^2$

$= 4\log_3 3 - 2\log_3 3$

$= 4 - 2 = 2$

2.4 Logarithm to base 10

Logarithms in base 10 are called **common logarithm** as they are used in many common scales, such as the **Richter scale** for measuring the magnitude of earthquakes, the **pH scale** for measuring acidity or alkalinity and the **decibel scale** for measuring sound.

$\log_{10} x$ is often written as just log x, and we *assume* the logarithm has base 10.

Rules of common logarithms:

The rules of common logarithm given below are similar to the one derived earlier:

- $\log(xy) = \log x + \log y$
- $\log\left(\frac{x}{y}\right) = \log x - \log y$
- $\log\left(x^n\right) = n \times \log x$
- $\log 1 = 0$
- $\log 10 = 1$

Example 6: Express the following as a single logarithm

(a) $\log 2 + \log 7$ (b) $\log 6 - \log 3$

Solution:

(a) $\log 2 + \log 7 = \log (2 \times 7)$

$= \log 14$

(b) $\log 6 - \log 3 = \log \frac{6}{3}$

$= \log 2$

Example 7: If $\log_3 7 = a$ and $\log_3 4 = b$. Write the following in terms of a and b.

(a) $\log_3\left(\frac{4}{7}\right)$ (b) $\log_3 28$ (c) $\log_3\left(\frac{7}{3}\right)$

Solution:

(a) $\log_3\left(\frac{4}{7}\right) = \log_3 4 - \log_3 7 = b - a.$

(b) $\log_3 28 = \log_3(4\times7) = \log_3 4 + \log_3 7 = b + a.$

(c) $\log_3\left(\frac{7}{3}\right) = \log_3 7 - \log_3 3 = a - 1.$

Example 8: Express the following as a single logarithm:

(a) $3 - \log_2 5$ (b) $1 + \log 2$

Solution:

(a) $3 - \log_2 5 = 3\times1 - \log_2 5$

Substituting $\log_2 2 = 1$ we get

$3 - \log_2 5 = 3\times\log_2 2 - \log_2 5$

$= \log_2 2^3 - \log_2 5$

$$= \log_2 8 - \log_2 5$$

$$= \log_2 \frac{8}{5}.$$

(b) $1 + \log 2 = \log 10 + \log 2$ (as $\log_{10} 10 = 1$)

$$= \log (10 \times 2)$$

$$= \log 20$$

Note:

- Common logarithms have an interesting property of scaling positive numbers that are very small or very large. For example if a certain quantity can take values from 0.0000000001 to 10,000,000,000 then the common logarithms of these numbers would lie in range –10 to 10.
- Natural logarithm are logarithm to the base e $(2 < e < 3)$, where e like π is an irrational number. The natural logarithm are denoted by ln i.e., $\log_e x = \ln x$.

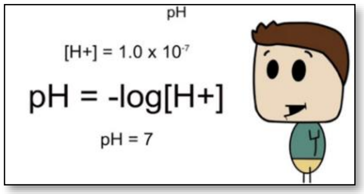
EXERCISE 2.2

1. Express the following as a single logarithm:
   - (a) $\log 2 + 2 \log 7$
   - (b) $\log_3 8 + \log_3 5 - \log_3 4$
   - (c) $\log 5 + 2 \log 3 - \log 15$
   - (d) $2 + 2\log_5 3$
   - (e) $3 - \frac{1}{2}\log_3 9$
   - (f) $1 + 2\log_4 3 - 3\log_4 4$
2. Find the exact value of
   - (a) $\log_{11} 121$
   - (b) $\log_7 1$
   - (c) $\log_5 625$
   - (d) $\log_8 8$
   - (e) $\log 1000$
3. If $\log_2 3 = p$ and $\log_2 5 = q$. Write the following in terms of p and q
   - (a) $\log_2 15$
   - (b) $\log_2 45$
   - (c) $\log_2 \left(\frac{5}{3}\right)$
   - (d) $\log_2 10$
4. Which of the following are true?
   - (a) If $2^{x+1} = 3^{x+2}$ then $x + 1 = x + 2$
   - (b) $\log (x + 1) = \log x$
   - (c) $\log_b b^3 = 3$
   - (d) Logarithm to base 1 is not defined.
5. If $\log_{2026} x - \log_{2026} y = a$, $\log_{2026} y - \log_{2026} z = b$ and $\log_{2026} z - \log_{2026} x = c$, then find the value of $\left(\frac{x}{y}\right)^{b-c} \times \left(\frac{y}{z}\right)^{c-a} \times \left(\frac{z}{x}\right)^{a-b}$.

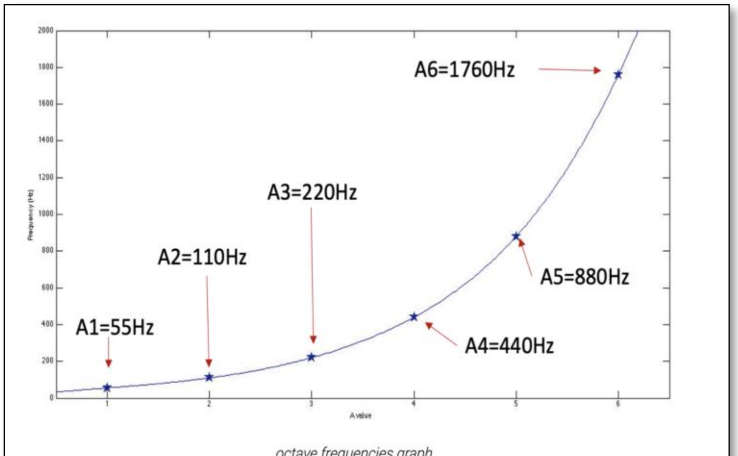
2.5 Logarithms Across Subjects

Logarithms are the "mathematical tools" used in Science, Music, and even Social science to manage scales that grow too fast such as population or earthquakes.

1. **Chemistry (The pH Scale):** Scientists measure how acidic a liquid is using the concentration of Hydrogen ions. Because these numbers are very small (like 0.00001), they use a negative log scale: $pH = -\log_{10}[H^+]$

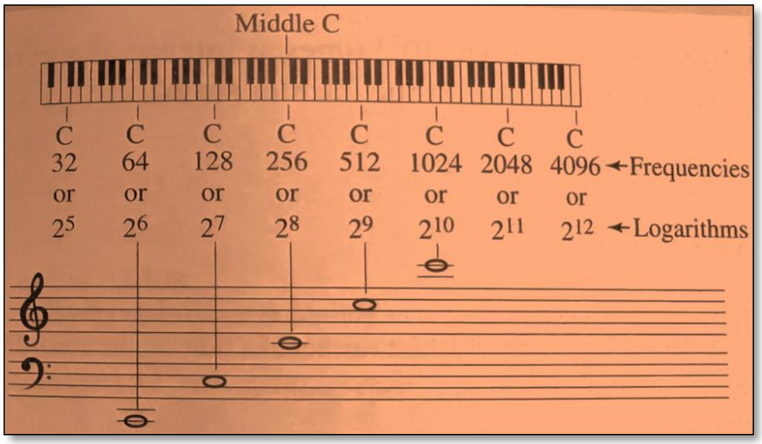


2. **Music (The Octave):** When a musician goes up one octave, the frequency of the sound doubles (2^1, 2^2, 2^3) but we perceive it as equal steps (1, 2, 3). It is interesting to learn that our ears hear sound logarithmically.



octave frequencies graph

The above graph shows all the A-note octaves from A1 to A6.



Note that frequency of each C note is written as powers of 2.

3. **Social Science:** Population growth often follows exponential patterns: $P = P_0(1+r)^t$. Logarithms help calculate **time required for population to double**.

4. **Geography (Earthquakes):** The Richter scale used to measure earthquakes is logarithmic. A magnitude 7 earthquake is not "two units" stronger than a magnitude 5 earthquake; it is 10^2 (100 times) more intense in amplitude!

Example 9: An earthquake measures 2 on the Richter scale, while another earthquake measures 5 on the same scale. How many times stronger is the second earthquake than the first?

Solution: Richter scale difference:

$$5 - 2 = 3$$

On the Richter scale, each 1-unit increase means **10 times stronger**.

So a 3-unit increase means:

$$10^3 = 1000$$

The magnitude 5 earthquake is **1000 times** stronger than the magnitude 2 earthquake.

EXERCISE 2.3

1. Express the following in logarithmic form:

   (a) $5^4 = 625$ (b) $10^{-2} = 0.01$

   (c) $7^0 = 1$ (d) $8^1 = 8$

2. Using the properties of logs, simplify: $\log_2 16 + \log_2 4$

3. Evaluate:

   (a) $\log_2 256$ (b) $\log_4 16$

   (c) $\log_5 125$ (d) $\log_{10} 0.001$

4. If $\log_2 7 = p$ and $\log_2 3 = q$. Write in terms of p and q.

   (a) $\log_2 21$ (b) $\log_2 49$

   (c) $\log_2\left(\frac{7}{3}\right)$ (d) $\log_2 63$

5. Real-world Application:

   (a) If a star is 100 times brighter than another, and the difference in their magnitudes is given by $2.5 \times \log_{10}$(brightness ratio), find the magnitude difference.

(b) A solution has pH 3 and another has pH 6. How many times more acidic is the first solution? $(\text{pH} = -\log_{10}[\text{H}^+])$

(c) A magnitude 9 earthquake occurs on the Richter Scale. How many times stronger is it than a magnitude 4 earthquake?

6. True or False: (Explain your reasoning)

(a) $\frac{1}{3}\log_b x = \sqrt[3]{x};\ x > 0$

(b) $\log_8 e = \frac{1}{\ln 8}$

(c) Logarithm of a negative number is defined.

(d) $\log_b(\text{M}+\text{N}) = \log_b \text{M} + \log_b \text{N}$

(e) The base of the logarithm can be any real number.

7. Which is the greatest integer that is less than the number $\log_4 9 + \log_9 28$? (Do not use calculator)

8. Evaluate the value of $(x + 5y)$, where, $x = \log_{1.4\overline{3}}\left(\frac{43}{40}\right)$ and $y = \left(\frac{1}{2}\right)^{\log_2 5}$.

2.6 Solving Logarithmic Equations: The Search for '*x*'

In our journey through logarithms, we have seen how they help us rethink the relationship between numbers and exponents. However, the true power of a logarithm is revealed when it becomes an active tool for solving equations where the unknown value, x, is trapped within a logarithm or a base.

Solving a logarithmic equation is much like being a mathematical detective. You must use the Product, Quotient, and Power rules to combine multiple logarithms into a single expression, and then convert it into its exponential form to find the value of the variable.

The Golden Rule of Logarithmic Equations: You must always verify your solutions! Because the domain of a logarithmic function is strictly positive, the value of $\log_b a$ is defined if a is positive ($a > 0$), and base (b) must be positive and not equal to 1 ($b > 0$, $b \neq 1$). Sometimes, standard algebraic steps will produce an **extraneous root**—a false solution that mathematically breaks these rules. If substituting your answer back into the original equation results in the logarithm of a negative number or zero, that solution must be rejected.

We will understand this through the following solved examples.

Example 10: Solve for $x : \log_2(3x - 1) = 3$.

Solution: $\log_2(3x - 1) = 3$

Converting logarithmic equation into its exponential form,

$$3x - 1 = 2^3$$

$$\Rightarrow \quad 3x = 9$$

$$\Rightarrow \quad x = 3$$

Check: Substitute $x = 3$ in $\log_2(3x - 1)$ gives $\log_2 8$. Since $8 > 0$, the logarithm is defined.

$\therefore$ $x = 3$ is the required solution.

Example 11: Solve for $x : \log_5 x + \log_5(x - 4) = 1$

Solution: Using the product rule, we get

$$\log_5[x(x - 4)] = 1$$

$$\Rightarrow \quad x(x - 4) = 5^1$$

$$\Rightarrow \quad x^2 - 4x - 5 = 0$$

$$\Rightarrow \quad (x - 5)(x + 1) = 0$$

$$\Rightarrow \quad x = 5, -1$$

Note that on substituting $x = -1$, in the equation gives $\log_5(-1)$ and $\log_5(-5)$. Since log of a negative number is not defined so $x = -1$, is an extraneous root which is rejected.

The only valid solution is $x = 5$.

Example 12: Solve for $x : \log_3(x^2 - 8x) = 2$

Solution: $\log_3(x^2 - 8x) = 2$

Converting logarithmic equation into its exponential form,

$$x^2 - 8x = 3^2$$

$$\Rightarrow \quad x^2 - 8x - 9 = 0$$

$$\Rightarrow \quad (x - 9)(x + 1) = 0$$

$$\Rightarrow \quad x = 9, -1$$

Note that for both $x = 9, -1$, the expression $(x^2 - 8x)$ evaluates to 9, which is strictly greater than zero. Therefore, both $x = 9, -1$ are valid solutions.

Example 13: Solve for $x : \log_2(3x-4) = \log_2 5$

Solution: $\log_2(3x-4) = \log_2 5$

$\Rightarrow \quad 3x - 4 = 5 \qquad$ (as $\log_a x = \log_a y \Rightarrow x = y$)

$\Rightarrow \quad 3x = 9$

$\Rightarrow \quad x = 3$

Example 14: Solve for $x : (\log_3 x)^2 - 5(\log_3 x) + 6 = 0$

Solution: $(\log_3 x)^2 - 5(\log_3 x) + 6 = 0$

Let $y = \log_3 x,$ the given equation reduces to

$$y^2 - 5y + 6 = 0$$

$\Rightarrow \quad (y-3)(y-2) = 0$

$\Rightarrow \quad y = 3$ or $y = 2$

$\therefore \quad \log_3 x = 3$ or $\log_3 x = 2$

$\Rightarrow \quad x = 3^3$ or $x = 3^2$

i.e. $\quad x = 27, 9$

For both $x = 27$, and $x = 9$, the value of $\log_3 x$ is defined as $x > 0$, so required solutions are $x = 9, 27$.

Example 15: Solve for $x : \log_b(\log_b Ax) = 1;\ A > 0$

Solution: $\log_b(\log_b Ax) = 1$

$\Rightarrow \quad \log_b Ax = b$

$\Rightarrow \quad Ax = b^b$

$\Rightarrow \quad x = \frac{1}{A}(b^b)$

EXERCISE 2.4

1. Solve for x.
   - (a) $\log_3(2x-5) = 2$
   - (b) $\log_7(3x) + \log_7 2 = \log_7 24$
   - (c) $\log_5(x+3) - \log_5(x-1) = 1$
   - (d) $\log_2(x^2-7) = 3$
2. Solve for x.
   - (a) $\log_2(x-3) + \log_2(x+1) = 5$
   - (b) $2\log_4 x = \log_4(5x-4)$
   - (c) $\log_5(x+2) + \log_5(x-2) = 1$
   - (d) $\log_{10}(x-2) + \log_{10}(x+1) = 1$

3. Solve for x.

(a) $\log_x(3x+10) = 2$, where $x > 0$ and $x \neq 1$

(b) $(\log_3 x)^2 - 4\log_3 x + 3 = 0$

(c) $(\log_2 x)^2 + \log_2 x^3 = 10$

(d) $x^{\log_{10} x} = 1000x^2$

4. Solve for x.

(a) $\log_3(x^2-1) = \log_3(2x-1)$

(b) $\log_x 5 - \log_x 2 = \log_x \sqrt{x}$

(c) $\log_2 x + \dfrac{1}{\log_x 2} = 4$

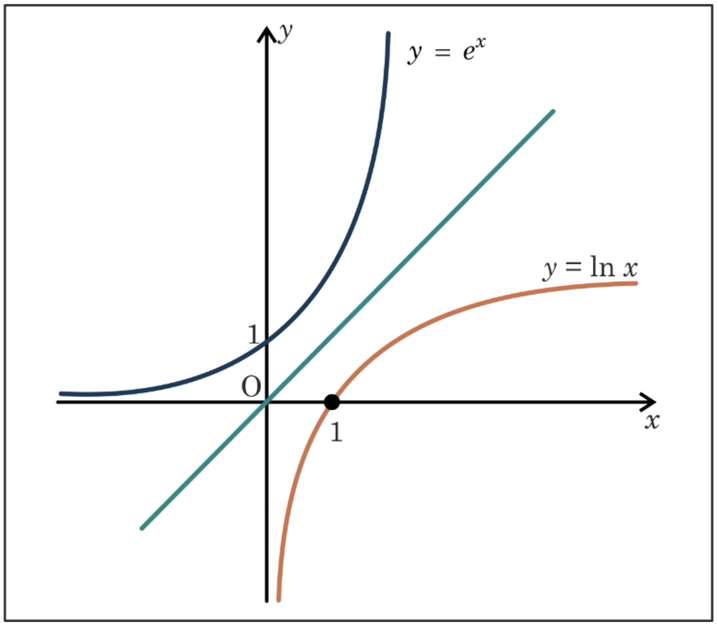
(d) $\log_3(3+x)+\log_3(8-x)-\log_3(9x-8) = 2-\log_3 9$

(e) $\log_{10}\left[\log_2\left(\log_3 9\right)\right] = 5x$

5. If $x = \log\dfrac{1}{2}+\log\dfrac{2}{3}+\log\dfrac{3}{4}+\ldots+\log\dfrac{99}{100}$, where all logs are to the base 10, then evaluate $(x + 1)\ (x + 2)\ (x + 3)\ \ldots\ (x + 99)$.

Enrichment – Graph of logarithmic and Exponential functions

The graph of logarithmic function and exponential function are inverse of each other. Note that the graph of these functions are mirror images along the line $y = x$.



Summary

- **Logarithm** is the inverse operation of exponentiation.
- $\log_b a = x \Leftrightarrow b^x = a$, where $a > 0$ and $b > 0$, $b \neq 1$
- Logarithms are defined only for positive numbers.
- If the base of a logarithm is not given then we assume it to be 10 and such logarithms are called common logarithms.
- **Important Properties:**

  For $b > 0$, $b \neq 1$ and M > 0, N > 0 we have:

  - Product Rule: $\log_b(M \times N) = \log_b M + \log_b N$
  - Quotient Rule: $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$
  - Power Rule: $\log_b(M^k) = k \times \log_b M$
  - Log of a number to the same base: $\log_b(b) = 1$
  - Log of 1: $\log_b(1) = 0$
  - The base changing property:

    For any logarithmic bases b and a and any positive number n,

    $$\log_b n = \frac{\log_a n}{\log_a b}$$

- **Applications:** Logarithms are widely used in:
  - Earthquake measurement
  - Acidity (pH scale)
  - Sound intensity
  - Population growth
  - Sports science